TRINITY COLLEGE

Mathematics Methods Units 3,4
Test 1 2017

Section 1 Calculator Free

Differentiation, Applications of Differentiation, Anti Differentiation

STUDENT'S NAME SOLUTIONS

DATE: Thursday 2 March **TIME:** 33 minutes **MARKS**: 33

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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1. (6 marks)

Given $y = x + \sqrt{x^2 - 4}$, show that $(x^2 - 4)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} - y = 0$

$$y' = 1 + \frac{2x}{2(x^2-4)^{\frac{1}{2}}}$$

$$= 1 + \frac{x}{(x^2-4)^{\frac{1}{2}}}$$

$$y'' = \frac{(x^2-4)^{\frac{1}{2}} - x \cdot \frac{1}{\cancel{2}}(x^2-4)^{-\frac{1}{2}} \cdot \cancel{2}x}{x^2-4}$$

$$(x^2-4)\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$$

$$= \frac{\cancel{(x^2-4)}(x^2-4)^{\frac{1}{2}}}{\cancel{(x^2-4)}} - \frac{\cancel{(x^2-4)}x^2(x^2-4)^{-\frac{1}{2}}}{\cancel{(x^2-4)}} + \cancel{x} + x^2(x^2-4)^{-\frac{1}{2}} - \cancel{x} - (x^2-4)^{\frac{1}{2}}$$

$$= \cancel{(x^2-4)^{\frac{1}{2}}} - \cancel{x^2(x^2-4)^{-\frac{1}{2}}} + \cancel{x^2(x^2-4)^{-\frac{1}{2}}} - \cancel{(x^2-4)^{\frac{1}{2}}}$$

$$= 0$$

2. (5 marks)

Use calculus to determine the % error in the volume of a spherical hot air balloon of diameter 32 metres if no allowance was made for the stretching of the material resulting in a 3% error in the diameter.

$$\frac{\delta r}{r} = 0.03$$

$$\frac{\delta v}{V} \approx \frac{dv}{dr} \cdot \frac{\delta r}{V}$$

$$\approx \cancel{4\pi} r^2 \cdot \frac{\delta r \, 3}{\cancel{4\pi} r^3}$$

$$\approx 3 \frac{\delta r}{r}$$

$$\approx 3 \times 0.03$$

$$\approx 0.09 \qquad \text{ie } 9\% \text{ INC}$$

3. (10 marks)

Determine each of the following.

(a) $\int \frac{2x - x^5}{3x^4} dx$ [3]

$$= \int \left(\frac{2}{3x^3} - \frac{x}{3}\right) dx$$

$$= \int \left(\frac{2x^{-3}}{3} - \frac{x}{3}\right) dx$$

$$= \frac{2x^{-2}}{-6} - \frac{x^2}{6} + c \qquad = -\frac{1}{3x^2} - \frac{x^2}{6} + c$$

(b) $\int \frac{2}{\sqrt{1-2x}} dx$ [3]

$$= \int 2(1-2x)^{-\frac{1}{2}} dx$$

$$= \frac{2(1-2x)^{\frac{1}{2}}}{\frac{1}{2} \cdot (-2)} + c$$

$$= -2(1-2x)^{\frac{1}{2}} + c$$

(c) $\int_{-1}^{2} (x-2)^2 dx$ [4]

$$= \int_{-1}^{2} (x^2 - 4x + 4) dx$$

$$= \left[\frac{x^3}{3} - 2x^2 + 4x\right]_{-1}^{2}$$

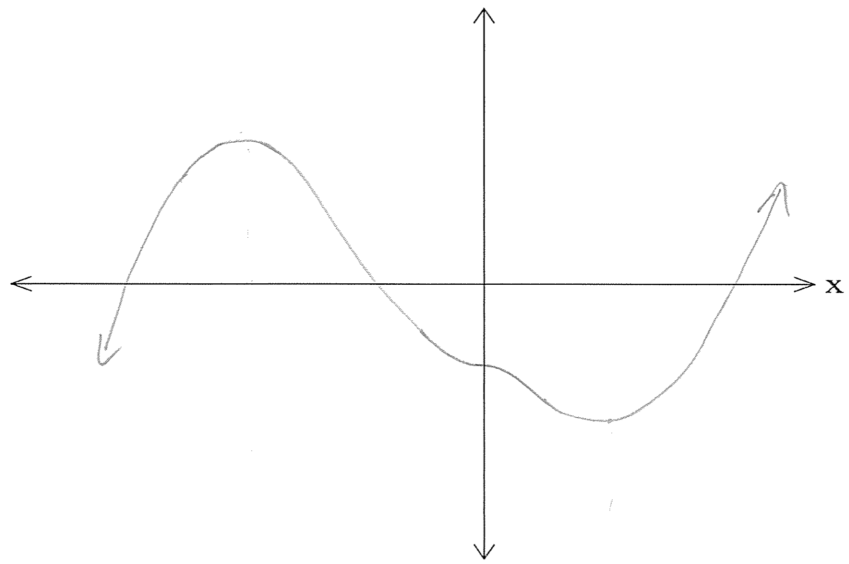
$$= \left(\frac{8}{3} - 8 + 8\right) - \left(-\frac{1}{3} - 2 - 4\right)$$

$$= 9$$

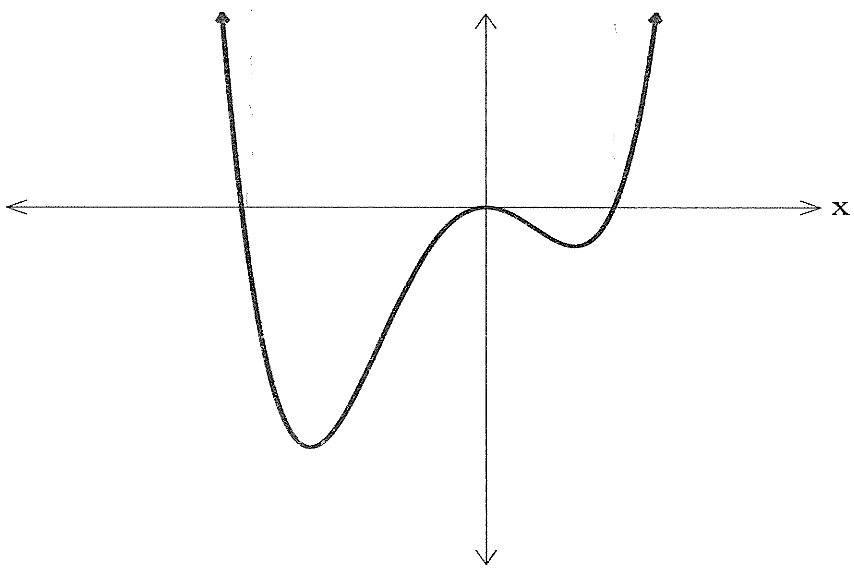
4. (6 marks)

Given the sketch of $y = f'(x)$, sketch $y = f(x)$ and $y = f''(x)$ below.

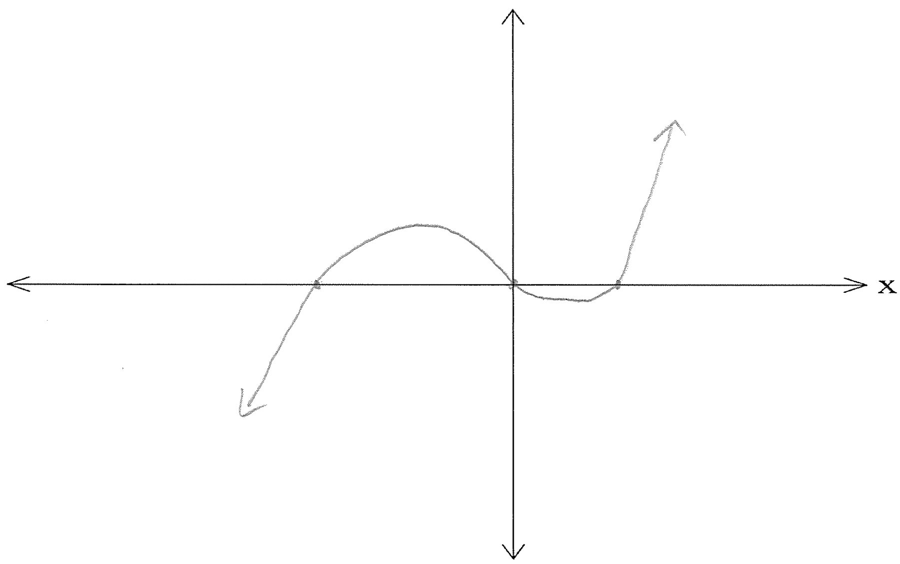
(a) $y = f(x)$ [3]



$y = f'(x)$



(b) $y = f''(x)$ [3]



5. (6 marks)

By determining each of the following

- Stationary points
- Points of inflection
- Axis intercepts
- Values of y for $x \to \pm\infty$

sketch $y = -x^3 - 3x^2 + 4$ on the axes below.

$y' = -3x^2 - 6x$

$-3x^2 - 6x = 0$

$-3x(x+2) = 0$

$x = 0, -2$

$(0, 4), (-2, 0)$

TP

$y'' = -6x - 6$

$-6x - 6 = 0$

$x = -1$

$(-1, 2)$ PT INFLECTION

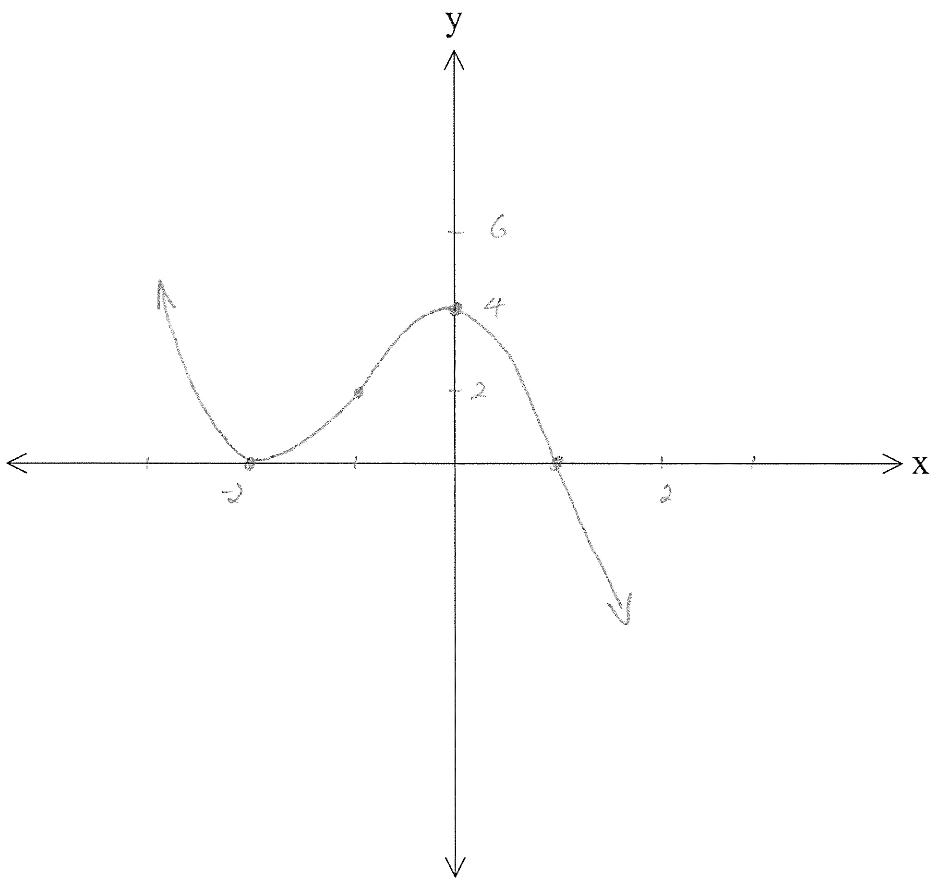
$x = 0$ $y = 4$ $(0, 4)$

$y = 0$ $x = 1$ BY OBSERVATION

$y = 0$ $x = -2$ (TP)

$x \to \infty$ $y \to -\infty$

$x \to -\infty$ $y \to \infty$



TRINITY COLLEGE

Mathematics Methods Units 3,4
Test 1 2017

Section 2 Calculator Assumed

Differentiation, Applications of Differentiation, Anti Differentiation

STUDENT'S NAME ______________________________

DATE: Thursday 2 March **TIME:** 21 minutes **MARKS**: 21

INSTRUCTIONS:

Standard Items: Pens, pencils, drawing templates, eraser

Special Items: Three calculators, notes on one side of a single A4 page (these notes to be handed in with this assessment)

Questions or parts of questions worth more than 2 marks require working to be shown to receive full marks.

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6. (4 marks)

The point $(2, b)$ lies on $y = \dfrac{a+4x}{3x+5}$ and the gradient at that point is 8. Determine a and b.

$$y' = \frac{4(3x+5) - 3(a+4x)}{(3x+5)^2}$$

$x = 2$
$m = 8$

$$8 = \frac{44 - 3a - 24}{121}$$

$$968 = 20 - 3a$$

$$3a = -948$$

$$a = -316$$

$(2, b)$

$$b = \frac{-316 + 8}{11}$$

$$= -\frac{308}{11}$$

$$= -28$$

7. (4 marks)

The duration of one vibration of a pendulum of length l is given by $t = \pi\sqrt{\frac{l}{1.1}}$, where t is measured in seconds and l is measured in centimetres. Given that a pendulum of length 97.8 cm vibrates once a second, use calculus to determine the approximate change in time of one vibration if the pendulum is lengthened to a metre.

$t = \frac{\pi \ell^{\frac{1}{2}}}{\sqrt{1.1}}$

$\delta \ell = 2.2$

$\frac{dt}{d\ell} = \frac{\pi}{2\ell^{\frac{1}{2}}\sqrt{1.1}}$

$\delta t \simeq \frac{dt}{d\ell} \times \delta\ell$

$\approx \frac{\pi}{2\ell^{\frac{1}{2}}\sqrt{1.1}} \times 2.2$

$(\ell = 97.8)$

$= 0.33$ sec

8. (4 marks)

During the course of an epidemic, the proportion of the population infected t months after the Epidemic began is given by $p = \frac{t^2}{5(1+t^2)^2}$.

(a) Determine the maximum proportion of the population that becomes infected. [2]

0.05

(b) Determine the time at which the proportion infected is increasing most rapidly. [2]

0.36

9. (4 marks)

Determine an expression for $f(x)$ if $f'(x) = x^2 + x + k$ for all x and $f(0) = -2$ and $f(-1) = 0$

$$f(x) = \int (x^2 + x + k)\, dx$$

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} + kx + c$$

$f(0) = -2$

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} + kx - 2$$

$f(-1) = 0$

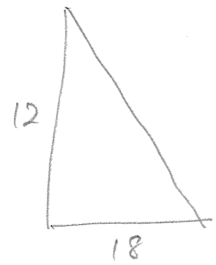
$$0 = -\frac{1}{3} + \frac{1}{2} - k - 2$$

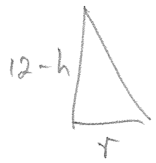
$$k = -\frac{11}{6}$$

$$f(x) = \frac{x^3}{3} + \frac{x^2}{2} - \frac{11x}{6} - 2$$

10. (5 marks)

A right circular cone has a radius of 18 cm and a height of 12 cm. Determine the volume of the largest cylinder which will fit inside the cone.

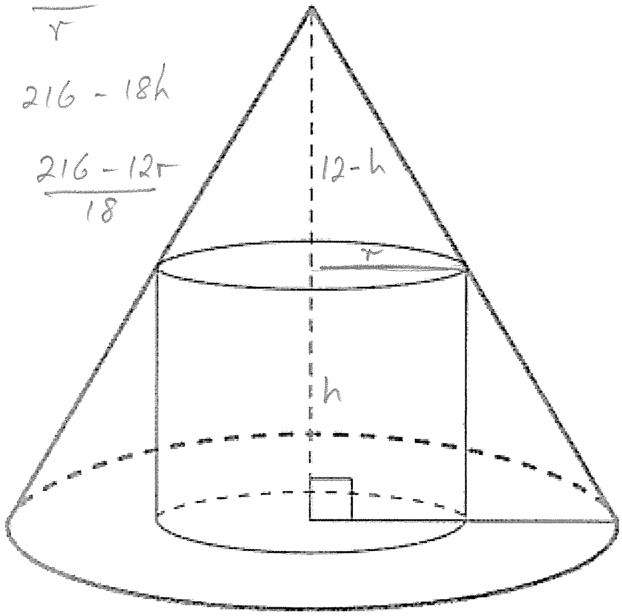




$$\frac{12}{18} = \frac{12-h}{r}$$

$$12r = 216 - 18h$$

$$h = \frac{216-12r}{18}$$



$$V = \pi r^2 h$$

$$= \pi r^2 \left(\frac{216 - 12r}{18}\right)$$

MAX WHEN $r = 12$

VOL $= 1809.6 \text{ cm}^3$